Math 10B with Professor Stankova
Worksheet, Midterm #1; Friday, 2/16/2018
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True/False

1. **TRUE** False Among the problems we considered in class, a multi-stage process can be encoded (and solved) by either dependent choice or independent choice at each stage, or split into mutually exclusive cases.
2. True **FALSE** We can turn any counting problem into a problem using the product rule or the sum rule.
3. True **FALSE** Reversing the order of stages in a process does not affect the difficulty or efficiency of solving the problem.
4. True **FALSE** $A \times B \times C$ for some sets A, B, C is another set made of all possible triplets (x, y, z) where x, y, z are any elements of the three sets.

Solution: We need $x \in A, y \in B, z \in C$.

5. **TRUE** False We solved in class the problem of finding the size of the power of a set by setting up a multi-stage process with 2 independent choices at each stage.

Solution: Each stage was deciding whether or not to put an element in the subset or not.

6. **TRUE** False The product rule for counting usually applies if we use the word "AND" between the stages of the process, while the sum rule for counting is usually used when we can finish the whole process in different ways/algorithms and we use the word "OR" to move from one way to another.
7. True **FALSE** Among the problems we considered so far in class, a multi-stage process can be encoded (and solved) by either dependent choice or independent choice at each stage, split into mutually exclusive cases, or split into "good" and "bad" cases.

Solution: We can also use PIE.

8. **TRUE** False Counting problems where the phrase "at least once" appears may indicate using the complement, or equivalently, counting all cases and subtracting from them all "bad" cases.

9. **TRUE** False Tree diagrams present a visual explanation of a situation, but unless one draw the full tree diagram to take into account all possible cases, the problem is not solved and will need more explanation/justification.

10. True **FALSE** To find how many natural numbers $\leq n$ are divisible by d, we calculate the fraction n/d and round up in order to not miss any numbers.

Solution: We round the fraction down, not up.

11. **TRUE** False We use 1 more than the ceiling (and not the floor) function in the statement of the Most General PHP because, roughly, we want to have one more pigeon that the ratio of pigeons to holes in order to "populate" a hole with the desired number of pigeons.

12. True **FALSE** It is always true that $\lfloor x \rfloor \leq x \leq \lceil x \rceil$ for any real number x, but equality of the two extreme terms of this inequality is never possible.

Solution: There is equality when x is an integer.

13. **TRUE** False Proof by contradiction can be used to justify any version of the PHP.

14. **TRUE** False A phrase of the type "at least these many objects" indicates what the pigeons should be in a solution with PHP, while "share this type of property" points to what the holes should be and how to decide to put a pigeon into a hole.

Solution: The objects in "at least these many objects" should be the pigeons and properties in "share this type of property" should be the holes.

15. **TRUE** False Erdos-Szekeres Theorem on monotone sequences is a generalization of the class problem on existence of an increasing or a decreasing subsequence of a certain length, and its proof assumes that one of two possibilities is not happening and shows that the other possibility must then occur.

> **Solution:** This is how we proved that in a sequence of 10 people, there must be at least 4 in an increasing or decreasing order.

16. True **FALSE** Any version of the PHP implies existence of certain objects with certain properties and shows us how to find them.

> **Solution:** It doesn't tell us how to find them, only that it exists.

17. **TRUE** False To prove that there are some two points exactly 1 inch apart colored the same way on a canvas painted in black and white, it suffices to pick an equilateral triangle of side 1 in on this canvas and apply PHP to its vertices being the pigeons and the two colors (black and white) being the holes.

18. True **FALSE** To show that a conclusion does not follow from the given conditions, we need to do more work than just show one counterexample.

> **Solution:** All we need to do is to show one counterexample.

19. **TRUE** False A counterexample is a situation where the hypothesis (conditions) of a statement are satisfied but the conclusion is false.

20. True **FALSE** The k-permutations of an n-element set are a special case of the k-combinations of this set.

21. **TRUE** False An identity is an equality that is always true for any allowable values of the variables appearing in the equality.

22. **TRUE** False An ordered k-tuple can be thought of some permutation of k elements, while an unordered k-tuple can be thought of a combination of k elements (perhaps, coming from a larger set).

23. **TRUE** False To prove some identity combinatorially roughly means to count the same quantity in two different ways and to equate the resulting expressions (or numbers).

24. **TRUE** False One good reason for 0! to be defined as 1 is for the general formula with factorials for C(n,k) to also work for k=0.

25. True **FALSE** The number of combinations C(n,k) is the number of permutations P(n,k) divided by the number of permutations P(n,n).

Solution: It should be divided by P(k, k).

26. True **FALSE** The symmetry of permutations can be seen in the identity P(n,k)=P(n,n-k) for all integer $n, k \geq 0$.

Solution: There is a symmetry of combinations, not permutations.

27. True **FALSE** The number of ways to split 10 people into two 5-person teams to play volleyball is $\frac{10! \cdot 10!}{2}$ because forgetting the 2 in the denominator would result in an overcount by a factor 2, which can be interpreted as an additional assignment of a court to each team on which to play (not required by the problem!).

Solution: The answer is just $\binom{10}{5}$.

28. **TRUE** False It is possible to use Calculus to prove combinatorial identities.

Solution: We can take the derivative of $(1+x)^n$ to get the equality

$$1\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} = n2^{n-1}.$$

29. True **FALSE** Interpreting the same quantity in two different ways is not useful in proving binomial identities because, ultimately, one of the interpretations is harder (or impossible!) to calculate on its own.

Solution: Proving binomial identities through two interpretations is often the slickest way to do it.

30. **TRUE** False The binomial coefficients appear in Pascal's triangle, as coefficients in algebraic formulas, and as combinations.

31. True **FALSE** The alternating sum of the numbers in an even-numbered row of Pascal's triangle is zero for the simple reason that Pascal's triangle is symmetric across a vertical line; but the same statement for an odd-numbered row requires some deeper analysis since the numbers there do not readily cancel each other.

Solution: The alternating sum of an odd-numbered row is zero because of symmetry. Note that an odd numbered row has an even number of elements, for instance row 5 has the elements $\binom{5}{0}, \binom{5}{1}, \binom{5}{2}, \binom{5}{3}, \binom{5}{4}, \binom{5}{5}$.

32. **TRUE** False The basic combinatorial relation satisfied by binomial coefficients that makes it possible to identify all numbers in Pascal's triangle as some binomial coefficients can be written as $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$ for $n, k \geq 1$.

33. **TRUE** False The formula $1+2+3+\cdots+n = \binom{n+1}{2}$ for $n \geq 1$ is a special case of the Hockeystick Identity $\binom{k}{k}+\binom{k+1}{k}+\binom{k+2}{k}+\cdots+\binom{n}{k} = \binom{n+1}{k+1}$ for $n \geq k \geq 0$.

Solution: Take $k = 1$ and then we get

$$\binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \cdots + \binom{n}{1} = 1 + 2 + \cdots + n = \binom{n+1}{1+1} = \binom{n+1}{2}.$$

34. **TRUE** False The binomial coefficients first increase from left to right along a row in Pascal's triangle, but then they decrease from the middle to the end of the row.

35. True **FALSE** $k\binom{n}{k} = n\binom{n-1}{k-1}$ unless $k > n$.

Solution: This is true for all n, k, namely even when $k > n$. When $k > n$, then both sides are 0. For $k \leq n$, this is an application of Question 22 via

$$\binom{n}{k}\binom{k}{1} = \binom{n}{1}\binom{n-1}{k-1}.$$

Another way to think about this is creating a team of size k with a captain. First we can select the team and then a captain, or we can first select the captain and then select the rest of the team.

36. **TRUE** False The coefficient of x^3y^2 in $(x+y)^6$ is 0 because $2+3 \neq 6$; yet, it appears twice in the expanded form of $(x+y)^5$.

> **Solution:** The coefficient of x^3y^2 in $(x+y)^6$ is 0 since $2+3 \neq 6$. But, it appears two times in $(x+y)^5$, namely as $\binom{5}{2}x^2y^3$ and $\binom{5}{3}x^3y^2$.

37. **TRUE** False We can use the Binomial Theorem to prove all sorts of binomial identities, provided we recognize what x, y, and n to plug into it.

> **Solution:** We used the Binomial coefficient to prove that the sum of all of the elements in a row of the Pascal's triangle was 2^n by plugging in $x = y = 1$ and we proved that the alternating sum is 0 by plugging in $x = 1, y = -1$.

38. **TRUE** False In general, it is harder to handle balls-into-boxes problems where the function must be surjective than where the function is injective or there are no restrictions on it.
39. **TRUE** False The number of k combinations from n elements with possible repetition is $\binom{n+k-1}{n-1}$ and it matches the answer to the problem of distributing k identical biscuits to n hungry (distinguishable) dogs.
40. **TRUE** False The number of 7-letter English words (meaningful or not, with possible repetition of letters) is not equal to the ways to distribute 7 equal bonuses to 26 people (with possible multiple-bonus winners).
41. **TRUE** False The equation $x_1 + x_2 + x_3 + x_4 = 10$ in natural numbers has as many solutions as trying to feed 4 (different) dogs with 6 (identical) biscuits.

> **Solution:** Since we are working with natural numbers, each of the numbers has to be at least 1 so we can first feed each dog one biscuit and then we have 6 left to distribute.

42. True **FALSE** The expression $(x+y+z+t)^{2018}$ has $\binom{2020}{3}$ terms after multiplying through but before combining similar terms, and 4^{2018} terms after combining similar terms.

> **Solution:** It is 4^{2018} terms before combining similar terms and $\binom{2021}{3}$ terms after combining terms.

43. True **FALSE** When we solve a problem one way, it is not useful to try to solve it in a second way because we already did the problem.

44. **TRUE** False In general, it is harder to handle balls-into-boxes problems where the boxes are indistinguishable than where the boxes are distinguishable.

45. True **FALSE** The number of ways to distribute b distinguishable balls into u distinguishable urns is $u!S(b,u)$ and the answer was obtained by solving first the same problem with indistinguishable urns and then labeling (or coloring) the urns to make them distinguishable.

> **Solution:** That formula is if we need each urn to have at least one ball in it. If we don't have that requirement, the answer is u^b.

46. **TRUE** False The equation $x_1+x_2+x_3+x_4=10$ in natural numbers where the order of the variables does not matter has as many solutions as the number of ways to split 10-tuplets (10 identical kids) into 4 identical playpens, where each playpen has at least one kid.

47. **TRUE** False We need to add several Stirling numbers of the second kind in order to count the ways to distribute distinguishable balls to indistinguishable boxes because all situations split into cases according to how many boxes are actually non-empty.

48. True **FALSE** The direct formula for the Stirling numbers of the second kind can be derived using P.I.E., and this proof must be memorized in order to do well in this class.

> **Solution:** You do not need to memorize this proof.

49. **TRUE** False We can prove the recursive formula for the Stirling numbers in a way very similar to the basic binomial identity $\binom{n}{k}+\binom{n}{k+1}=\binom{n+1}{k+1}$ by selecting one special object and discussing the two possible cases from its viewpoint.

50. True **FALSE** Two problem types in the 12-fold way table have extremely simple answers because an injective function cannot have a smaller domain than co-domain.

51. True **FALSE** An algorithm is a finite sequence of well-defined steps that always lead to the correct (desired) output.

52. True **FALSE** The Quick Sort algorithm is, on the average, faster than the Bubble Sort algorithm because the number of inversions in the list being sorted increases faster during the Quick Sort algorithm.

Solution: The number of inversions decreases faster.

53. True **FALSE** In class, we defined an inversion as two adjacent elements in a list that are out of order.

Solution: It is defined as two elements, one appearing sometime before, not immediately before, another such that they are out of order.

54. True **FALSE** The maximal possible number of inversions in a list of 10 number is $P(10, 2) = 90$ while the minimal such number is 1.

Solution: The maximum is $\binom{10}{2}$ and the minimum is 0.

55. **TRUE** False A stable matching between n jobs and n people means that even if some person A prefers a job B on this list of jobs that he has not been given, the company that offers job B has hired another person C on it that they prefer to person A.

56. True **FALSE** The number of roommate pairings among 2018 people can be written as $2017 \cdot 2015 \cdot 2013 \cdots 3 \cdot 1$ or alternatively also as $\frac{2018!}{2^{2018} 1009!}$.

Solution: The latter formula should read $\frac{2018!}{2^{1009} 1009!}$.

57. True **FALSE** It is never possible to pair up 2018 people into stable roommate pairs because, even if we manage to pair up 2014 of them in stable pairs, there will always be 4 of them to produce a counterexample of an impossible stable pairing.

Solution: It is sometimes possible to pair them up. But we cannot always pair them up.

58. **TRUE** False To show that something is possible, it suffices to provide just one way of doing it, but to show that something is always true, we need to provide a proof that works for all cases.

59. True **FALSE** The stable marriage algorithm produces the same final stable pairing, even if we reorder the "men" or if we switch the places of "men" and "women".

> **Solution:** It may produce a different pairing if we reorder the men or switch the places. But, the pairing will always be stable.

60. True **FALSE** An argument by contradiction can be avoided if we are careful not to make mistakes in our proof.

61. True **FALSE** Within MMI, the inductive step "If S_n is true then S_{n+1} is also true." implies that S_{n+1} is true.

> **Solution:** We need S_n to be true in order for S_{n+1} to be true.

62. **TRUE** False When making the inductive hypothesis "Suppose S_n is true." we need to say "for some n "; yet, we cannot specify a particular number for n here.

63. True **FALSE** At the end of a successful application of MMI, we conclude that S_n is true for some particular n's.

> **Solution:** It is true for all n.

64. **TRUE** False If we do not know the final precise answer to a problem, we cannot apply MMI until we conjecture what this answer is.

65. **TRUE** False "Harry Potter is immortal." is not suitable for a proof by MMI, but it can be paraphrased into such a suitable statement.

> **Solution:** We need a series of statements. So, we could say that Harry Potter is alive on day n be a series of statements and if all of them are true, then Harry Potter is immortal.

66. True **FALSE** It is never necessary to show the first several base cases in a proof by MMI; indeed, we do this just to boost our confidence in the truthfulness of the statement of the problem and we need to show only that the first base case is true.

67. **TRUE** False Within the inductive step of a proof by MMI, we may occasionally need to use $S_{n-1}, S_{n-2}, S_{n-3}$, or some previous S_k (instead of S_n) in order to prove S_{n+1}.

68. True **FALSE** Since the world will never end, the Tower of Hanoi problem for 64 initial discs on one of the three poles cannot be solved, whether by MMI or other methods.

69. **TRUE** False Sending off newly-married couples to different honeymoon locations around the universe will provide a counterexample for the even version of the "Odd-pie fight" problem.

70. True **FALSE** The "complement" property of probabilities, $P(\overline{A}) = 1 - P(A)$ for any $A \subseteq \Omega$, should be added to the definition of the probability space (Ω, P) because it is fundamental and always works.

Solution: We can prove this from the definitions already there, so there is no need to state it as a rule.

71. **TRUE** False When calculating the probability $P(A)$ for some event $A \subseteq \Omega$ on an "equally likely" finite probability space (Ω, P) , we can simply count the number of outcomes of A (the good possibilities) and divide that by all outcomes in Ω (all possibilities).

72. True **FALSE** It is incorrect to say that the elements of Ω are "outcomes" since they are actually inputs of the probability P, and not outputs.

Solution: They are outcomes of the experiment and sets of them are inputs to the probability function P.

73. True **FALSE** The probability function P is defined as $P : \Omega \to [0, 1]$ such that $P(\Omega) = 1, P(\emptyset) = 0$, and $P(A \cup B) = P(A) + P(B)$ for any disjoint subsets A and B of Ω.

Solution: The probability function goes from subsets of Ω, not just Ω.

74. True **FALSE** The formula $\lceil \frac{N}{d} \rceil$ appears when calculating the probability of a natural number $n \leq N$ to be divisible by d.

Solution: It is the floor, not ceiling.

75. True **FALSE** MMI is not really necessary to formally prove the property $P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1)+P(A_2)+\cdots+P(A_n)$ for pairwise disjoint A_i's because the property is quite intuitive and, to prove it, we can just apply over and over again the basic property of probabilities $P(A \cup B) = P(A) + P(B)$ for various non-overlapping $A, B \subseteq \Omega$.

Solution: We need to use MMI.

76. **TRUE** False If we experiment with throwing two fair dice and adding up the two values on the dice, and if we decide to represent the outcome space Ω as the set all of possibilities for the sum; i.e., $\Omega = \{2, 3, \ldots, 12\}$, then the corresponding probability P will not be the "equally likely" probability, making us reconsider the choice of the outcome space Ω in the first place.

77. **TRUE** False The property $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ for any $A, B \subseteq \Omega$ is true for any probability space (Ω, P) , but it can be proven using baby P.I.E. only when P is the "equally likely" probability on a finite outcome space Ω , while a more general argument is needed for other (Ω, P).

78. True **FALSE** The "defective dice" problem from the Discrete Probability handout does not have a unique solution; i.e., there is another second die that can yield, together with the original defective die, the same probabilities for the sums of the values on the two dice as two normal fair dice.

79. **TRUE** False In the Monty Hall Problem with n doors (for any $n \geq 3$) we should switch doors (after the host opens a non-winning door) because with this strategy the probability of winning is $\frac{n-1}{n(n-2)}$; however, for $n = 2$ this formula makes no sense and, moreover, half of the time the game itself is impossible to complete as designed for $n = 2$.

80. **TRUE** False It may not be possible to calculate $P(A \cap B)$ using just $P(A)$ and $P(B)$, but if we also know one of $P(A|B)$ or $P(B|A)$, we can do it!

81. True **FALSE** $P(A|B)$ can never be equal to $P(B|A)$ unless $P(A) = P(B)$.

Solution: You could have $P(A \cap B) = 0$ and $P(A|B) = P(B|A) = 0$ but $P(A) \neq P(B)$.

82. True **FALSE** The formula $P(A) = P(A|B) \cdot P(B) + P(A|\overline{B}) \cdot P(\overline{B})$ works for any events $A, B \subseteq \Omega$, as long as $P(B) > 0$.

> **Solution:** We need that $P(B) < 1$ as well.

83. **TRUE** False Despite the suggestive notation, the conditional probability $P(A|B)$ was originally defined through a formula and we had to prove that it indeed is in $[0, 1]$ in order to consider $P(A|B)$ as an actual probability.

84. **TRUE** False To prove the Probability "Baby P.I.E." property $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, one could split $A \cup B$ on the LHS into three disjoint subsets, similarly split on the RHS A and B each into two subsets, cancel, and match the resulting probabilities on the two sides.

85. **TRUE** False When considering conditional probability, we are restricting the original outcome space Ω to a smaller subspace B given by the condition of something having happened.

86. **TRUE** False When $A \subset B$, the conditional probability $P(A|B)$ can be expressed as the fraction $\frac{P(A)}{P(B)}$ (given all involved quantities are well-defined).

> **Solution:** Since $A \subset B$, we know that $A \cap B = A$ and hence
>
> $$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}.$$

87. **TRUE** False Bonferroni's inequality $P(E \cap F) \geq p(E) + P(F) - 1$ is, in disguise, the well-known fact that $P(E \cup F) \leq 1$.

88. **TRUE** False When selecting at random two cards from a given 6-card hand (from a standard deck) that is known to contain 2 Kings, it is more likely to end up with at least one King than no King.

89. **TRUE** False When selecting at random two cards from a given n-card hand (from a standard deck) that is known to contain 2 Kings, the smallest n for which it is more likely to end up with no King than with at least one King is $n = 8$.

1 Problems

90. How many ways can you rearrange the letters in BERKELEY?

> **Solution:** There are 8 letters but 3 E's and no other letters repeat. So the answer is $\frac{8!}{3!}$.

91. There are 72 students trying to get into 3 of my sections. There are $27, 20, 25$ openings respectively. How many ways are there for these students to enroll?

Solution: $\binom{72}{27}\binom{72-27}{20}\binom{72-27-20}{25} = \binom{72}{27}\binom{45}{20}$.

92. How many ways can I put 20 Tootsie rolls into 5 goodie bags so that each goodie bag has at least 2 Tootsie roll?

Solution: Put one in each then there are 15 rolls left and 5 bags and everything is indistinguishable so $p_5(15)$.

93. Show that when you place 9 coins on an 8×10 boards, at least two coins must be on the same row.

Solution: PP, $9/8 > 1$.

94. How many license plates with 3 digits followed by 3 letters do not contain the both the number 0 and the letter O (it could have an O or a 0 but not both).

Solution: Complementary counting. Bad cases are if it has an O and a 0. There are $26^3 - 25^3$ ways to have a O, there are $10^3 - 9^3$ ways to have a 0. So the final answer is

$$10^3 \cdot 26^3 - (10^3 - 9^3)(26^3 - 25^3).$$

95. Prove that $\sum_{k=0}^{n} 5^k \binom{n}{k} = 5^0\binom{n}{0} + 5^1\binom{n}{1} + \cdots + 5^n\binom{n}{n} = 6^n$.

Solution: Use the Binomial theorem and plug in $x = 1, y = 5$.

96. How many ways can I split up 30 distinguishable students into 6 groups each of size 5?

Solution: $\frac{30!}{5!5!5!5!5!5!}$.

97. Find a formula for $1 + 2 + 4 + \cdots + 2^n$ and prove it.

Solution: The answer is $2^{n+1} - 1$ and use induction to prove it.

98. How many 5 digit numbers have strictly increasing digits (e.g. 12689 but not 13357).

Solution: $\binom{9}{5}$.

99. How many 5 digit numbers have increasing digits (you can have repeats e.g. 12223 or 22222)?

Solution: $\binom{13}{5}$.

100. A 7 phone digit number $d_1d_2d_3 - d_4d_5d_6d_7$ is called memorable if $d_1d_2d_3 = d_4d_5d_6$ or $d_1d_2d_3 = d_5d_6d_7$. How many memorable phone numbers are there?

Solution: We use PIE. There are $10^3 \cdot 10$ for each way and then we need to subtract the intersection. If $d_1d_2d_3 = d_4d_5d_6 = d_5d_6d_7$, then $d_4 = d_5 = d_6 = d_7 = d_1 = d_2 = d_3$ and so the intersection is if its a phone number with everything repeating. Therefore, there are 10 numbers in the intersection. This gives a total of

$$2 \cdot 10^3 \cdot 10 - 10$$

ways.

101. How many people do you need in order to guarantee that at least 3 have the same birthday?

Solution: $366 * 2 + 1$.

102. How many 5 letter words have at least two consecutive letters are the same?

Solution: We need to use complementary counting. The bad ways is if there are no two consecutive letters the same. There are $26 \cdot 25^4$ ways to do this. So the answer is $26^5 - 26 \cdot 25^4$.

103. Prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for all $n \geq 1$.

Solution: Induction.

104. How many ways can I put 60 seeds in 10 indistinguishable boxes so that each box has at least 3 seeds?

Solution: First put two seeds in the 10 boxes which leaves 40 left and each box needs to have at least one. So there are $p_{10}(40)$ ways.

105. I have 4 indistinguishable blue coins and 4 indistinguishable gold coins. How many ways can I stack them?

Solution: $\binom{8}{4}$.

106. When I go to CREAM, I order 4 scoops of ice cream out of 10 possible flavors (I can get more than one scoop of a flavor)?

Solution: $\binom{13}{4}$.

107. Show that in a class of 30 students, there must exist at least 10 freshmen, 8 sophomore, 8 juniors, or 7 seniors.

Solution: PP. $30 > (10-1) + (8-1) + (8-1) + (7-1) = 29$.

108. How many ways can I buy 24 donut holes if there are 8 different flavors?

Solution: $\binom{31}{7}$.

109. How many ways can I split 200 indistinguishable donut holes into 8 non-empty bags? (The bags are indistinguishable)

Solution: $p_8(200)$.

110. How many ways can I split 200 indistinguishable donut holes into at most 8 bags? (The bags are indistinguishable)

Solution: $p_1(200) + p_2(200) + \cdots + p_8(200)$.

111. I have 5 identical rings that I want to wear at once. How many ways can I put them on my hand (10 fingers) if each ring must go on a different finger?

Solution: $\binom{10}{5}$.

112. Suppose that Alice chooses 4 distinct numbers from the numbers 1 through 10 and Bob chooses 4 numbers as well. What is the probability that they chose at least one number in common?

Solution: We do this via complementary counting. The bad case is when they do not choose any numbers in common. Once Alice chooses 4, there are 6 more numbers for Bob to choose so there are $\binom{6}{4}$ ways for him to choose different numbers, and $\binom{10}{4}$ total number of ways. Therefore, the probability is

$$1 - \frac{\binom{6}{4}}{\binom{10}{4}}.$$

113. How many positive integers less than $10,000$ have digits that sum to 9?

Solution: This is the same as solving $x_1 + x_2 + x_3 + x_4 = 9$ so $\binom{12}{9}$ ways.

114. I am planting 15 trees, 5 willow trees and 10 fir trees. How many ways can I do this if the two willow trees cannot be next to each other?

Solution: We can place the willow trees down and we have 6 spots next to them so we want to solve $x_1 + x_2 + \cdots + x_6 = 10$ but $x_2, x_3, x_4, x_5 \geq 1$ since the willow trees can't be next to each other. This gives us $\binom{11}{6}$. Another way to see this is if we place the 10 fir trees down and then there are 11 spots for the 5 willow trees, giving $\binom{11}{5}$ ways.

115. How many ways can we put 5 distinct balls in 20 identical bins?

> **Solution:** $S(5,1) + S(5,2) + S(5,3) + S(5,4) + S(5,5)$. This is tricky because we don't need the 20 at all since we cannot have 20 non-empty bins.

116. How many ways can I distribute 30 Snickerdoodle cookies and 20 chocolate chip cookies to 25 students if there is no restriction on the number of cookies a student gets (and some students can get none)?

> **Solution:** $\binom{30+25-1}{30} \cdot \binom{20+25-1}{20}$.

117. What is the coefficient of the term $a^{10}b^{20}c^{30}$ in $(2x + 3y + 4z)^{60}$?

> **Solution:** It is $\binom{60}{10}\binom{50}{20}2^{10} \cdot 3^{20} \cdot 4^{30}$.

118. What is the probability that a roll a sum of 9 with two dice given that I rolled a 6?

> **Solution:** There are 4 ways to roll a 9 and a 6 namely $\{(3,6),(6,3)\}$ and 11 ways to roll at least one 6. So the probability is $\frac{2}{11}$.

119. In the US (300 million people), everyone is a male or female and likes one of 10 different colors. Show that there exist at least 3 people that have the same gender, like the same color, have the same three letter initial, and have the same birthday?

> **Solution:** PP. $300,000,000/(2 \cdot 10 \cdot 26^3 \cdot 366) > 2$.

120. I roll two die. What is the probability that I roll a 2 given that the product of the two numbers I rolled is even.

> **Solution:** First we want to count the number of ways I can roll an even number product. This is the total number of ways minus the number of ways I can roll and odd product, which is if I roll two odd numbers, which is $3^2 = 9$ ways. So there are 27 ways to roll an even product. Then the number of ones with a 2 are 11 if you list them out or note that either the first roll is a 2 or the second one is and the intersection has size 1 (2, 2). Therefore the answer is $\frac{11}{27}$.

121. I have 5 identical rings that I want to wear at once. How many ways can I put them on my hand (10 fingers) if it is possible to put all 5 on one finger?

Solution: $\binom{14}{5}$.

122. How many ways can I split up 30 distinct students into 6 non-empty groups?

Solution: $S(30, 6)$.

123. What is the probability that in a hand of 5 cards out of a deck of 52 cards, there is a pair of aces given that there is an ace?

Solution: There are $\binom{4}{2} \cdot \binom{48}{3}$ ways to have a pair of aces and there are $\binom{52}{5} - \binom{48}{5}$ ways to have at least one ace. So the conditional probability is

$$\frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5} - \binom{48}{5}}.$$

124. How many ways can I plant 15 trees in 5 different yards if each yard has to be nonempty?

Solution: $\binom{14}{10}$.

125. In a bag of coins, 2/3 of them are normal and 1/3 have both sides being heads. A random coin is selected and flipped and the outcome is a heads. What is the probability that it is a double head coin?

Solution: We have that

$$P(Double|Heads) = \frac{P(Heads|Double)P(Double)}{P(Heads|Double)P(Double) + P(Heads|Normal)P(Normal)}$$

$$= \frac{1 \cdot 1/3}{1 \cdot 1/3 + 1/2 \cdot 2/3} = \frac{1/3}{2/3} = \frac{1}{2}.$$

126. How many numbers less than or equal to 1000 are not divisible by $2, 3$, or 5?

> **Solution:** There are 500 divisible by 2, 333 divisible by 3, 200 divisible by 5. There are $1000/6 = 166$ divisible by 2 and 3 or 6, there are $1000/10 = 100$ divisible by 2 and 5, and $1000/15 = 66$ divisible by 3 and 5. There are $1000/30 = 33$ divisible by $2, 3, 5$. So the final answer is
>
> $$1000 - 500 - 333 - 200 + 166 + 100 + 66 - 33 = 266.$$

127. How many ways can you line up 4 couples if each couple needs to stand next to each other?

> **Solution:** 4! ways to order the 4 couples then 2! within each couple so $4! \cdot (2!)^4$.

128. Prove that $2 + 4 + \cdots + 2n = n(n + 1)$ for all $n \geq 1$.

> **Solution:** Induction.

129. How many ways can I put 30 Snickerdoodle cookies and 20 chocolate chip cookies into 10 identical bags so that each bag has at least one Snickerdoodle cookie and one chocolate chip cookie?

> **Solution:** $p_{10}(30) \cdot p_{10}(20)$.